UNIT-1

INTRODUCTION TO FINITE ELEMENT METHOD AND FUNCTIONAL APPROXIMATION METHOD

①

* Introduction:

The Finite element Method (FEM) is a numerical technique to find approximate solutions of partial differential equations. It was originated from the need of solving complex elasticity and structural analysis problems in Civil, Mechanical and Aerospace engineering.

In a structural simulation, FEM helps in producing stiffness & strength visualizations. It also helps to minimize material weight and its cost of the structures. FEM allows for detailed visualization and indicates the distribution of stresses and strains inside the body of a structure.

Several modern FEM packages include specific components such as fluid, thermal, electromagnetic, and structural working environments. FEM allows entire designs to be constructed refined and optimized before the design is manufactured.

→ Numerical Methods:

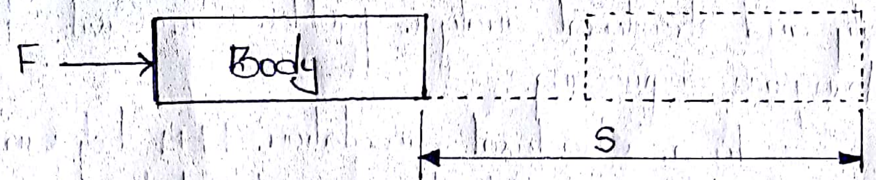
Numerical Methods which are commonly used to solve solid and fluid mechanics problems are given below.

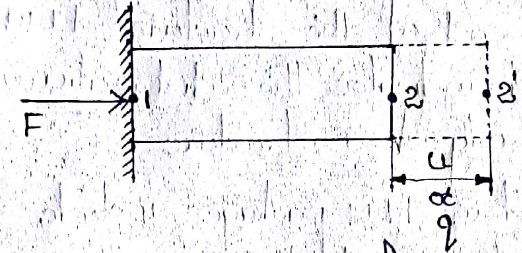
1. Finite Difference Method.
2. Finite Volume Method.
3. Finite Element Method.
4. Boundary element Method.
5. Meshless Method.

→ Concepts of elements and Nodes:

Any Continuum/Domain can be divided into number of pieces with very small dimensions. These small pieces/sub domains of finite dimension are called "Finite Elements". These Elements are connected through a number of joints which are called 'Nodes'. While discretizing the structural system, it is assumed that the elements are attached to the adjacent elements only at the nodal points. Each element contains the material and geometrical properties. The material properties inside an element are assumed to be constant. The elements may be 1D, 2D or 3D elements.

→ **Nodal Displacements :**

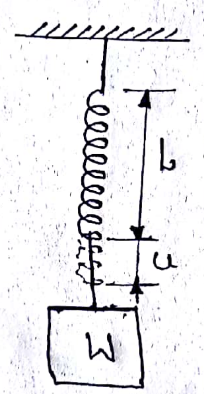




When the body is fixed at one Node, then we observe a Nodal displacement instead of body displacement.

Here, u = Nodal displacement

→ **String Element :**



$F \propto u$

$F = ku \longrightarrow (i)$

Where k = Stiffness

$\therefore k = F/u$.

We know that

$$\sigma = E\epsilon$$

$$\frac{F}{A} = E\frac{u}{L}$$

$$F = \frac{AE}{L}u \longrightarrow (ii)$$

$\therefore$ From (i) & (ii)

$$\therefore K = \frac{AE}{L}$$

→ Degree of Freedom (DOF):

Number of Components of the displacement vector at a Node.

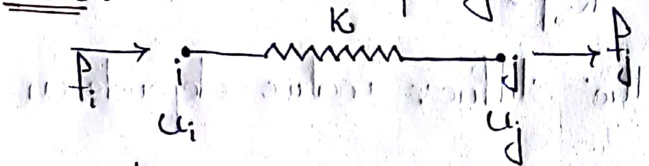
For Bar element : 2 dof.

i.e at Node 1, & 2.

At Node 1 = 1 dof.

2 = 1 dof, Total = 2 dof.

Note : Consider a Spring element.



Here, Nodes i, j

Disp. u_i, u_j

Forces f_i, f_j

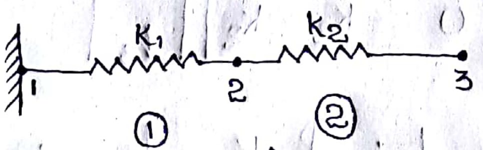
Stiffness K.

We have

$$[K] = \begin{bmatrix} K & -K \\ -K & K \end{bmatrix}$$

Where $[K]$ = Global stiffness Matrix for Spring element.

Problem : 1



Step 1 : Descretization of Domain

Elements	Nodes	
1	1	2
2	2	3

step 2 : Element stiffness Matrix.

$$[K_1] = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \quad \& \quad [K_2] = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}$$

Step 3: Global stiffness Matrix.

$$[K] = [K_1] + [K_2]$$

$$= \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array} + \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$$= \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Note:
Here, the size of the stiffness matrix depends on the no. of displacements.

Step 4: Considering Finite Element Equation.

$$F = ku$$

Where

F = Global Load Vector

K = Global stiffness Matrix.

U = Displacement Vector.

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

Step 5: Applying / Incorporation of Boundary Conditions.

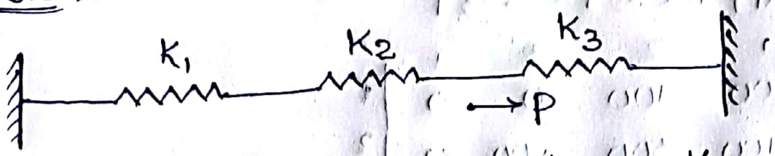
Since, Node 1 is fixed, $q_1 = 0$. So by using the Elimination Approach we eliminate 1st row & 1st column.

$$\begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

By solving the above Matrix, we calculate the displacements at a particular Node.

Problems:

1.



$K_1 = 100\ N/mm$ $K_2 = 200\ N/mm$ $K_3 = 100\ N/mm$

$P = 500N$

Find A) Global Stiffness Matrix.

B) Displacements

c) Reaction Forces.

Sol:- Step 1: Descretization of Domain.

Element	Nodes
①	1 2
②	2 3
③	3 4

Step 2: Calculating Element stiffness Matrix.

$$[k_1] = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{cc} 1 & 2 \end{array} & \\ \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$[k_2] = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{cc} 2 & 3 \end{array} & \\ \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$$[k_3] = \begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}$$

$$= \begin{array}{cc} \begin{array}{cc} 3 & 4 \end{array} & \\ \begin{bmatrix} 100 & -100 \\ -100 & 100 \end{bmatrix} & \begin{array}{c} 3 \\ 4 \end{array} \end{array}$$

Step 3: Calculating Global stiffness Matrix.

$$[K] = [K_1] + [K_2] + [K_3]$$

$$= \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

Step 4: Calculating Global load vector.

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} R_1 \\ 0 \\ 500N \\ 0 \end{bmatrix}$$

Step 5: Considering Finite Element Equation.

$$F = KU$$

$$\begin{bmatrix} R_1 \\ 0 \\ 500N \\ 0 \end{bmatrix} = \begin{bmatrix} 100 & -100 & 0 & 0 \\ -100 & 300 & -200 & 0 \\ 0 & -200 & 300 & -100 \\ 0 & 0 & -100 & 100 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Step 6: Incorporating Boundary Conditions.

Since Node 1 and Node 4 is fixed i.e., $q_1 = 0$. $q_4 = 0$. By Using Elimination Method, Eliminate 1st and 4th rows & columns.

$$\begin{bmatrix} 0 \\ 500 \end{bmatrix} = \begin{bmatrix} 300 & -200 \\ -200 & 300 \end{bmatrix} \begin{bmatrix} q_2 \\ q_3 \end{bmatrix}$$

$$300q_2 - 200q_3 = 0 \Rightarrow q_2 = \frac{200}{300} q_3 = 2/3\, q_3.$$

$$-200q_2 + 300q_3 = 500.$$

$$-200(2/3) + 300q_3 = 500$$

$$300q_3 = 500 + \frac{400}{3} = \frac{1500+400}{3} = \frac{1900}{3}.$$

$$q_3 = 2.11\,mm$$

$$\Rightarrow q_2 = 2/3(2.11) = 1.406\,mm.$$

Step 7: Calculating Reaction forces:

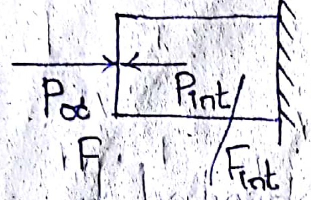
$$R_1 = -100q_2 = -140.67\,N.$$

$$R_4 = -100q_3 = -211\,N.$$

* **Stress :**

The Restoring force per unit area setup inside the body is known as stress.

$$\text{Stress} = \frac{\text{Restoring force}}{\text{Area}} = \frac{F}{A}$$



$\frac{F}{A}$ = Pressure

$\frac{F_{int}}{A}$ = Stress

Here F_{int} = Restoring force

Units : N/m^2

Types of stresses:

Stress
- Normal stress (Normal to Plain Surface)
- Volume or Bulk stress
- Tangential or Shear stress.

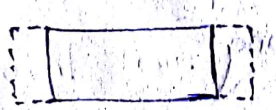
* **Strain :**

The ratio of change in dimension by Original dimension is known as strain.

$$\text{Strain} = \frac{\text{Change in dimension}}{\text{Original dimension}} = \frac{\Delta L}{L} \text{ (No Units).}$$

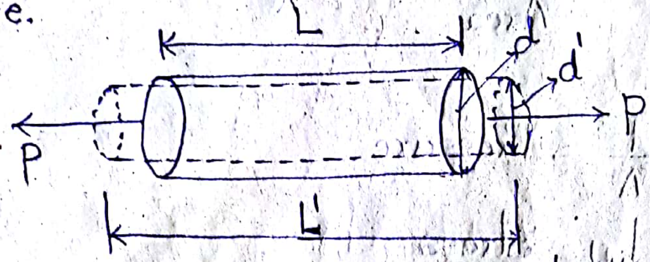
Types of strains:

Strain
- Normal strain → Size change
- Volume or Bulk strain: $\varepsilon_v = \frac{\Delta V}{V}$
- Shear strain → Shape change



Poisson's Ratio :

Consider a Rod of Length 'L' and diameter d. Let the Force 'P' applied in the axial direction as shown in figure.



Now, there is a change in length absorbed 'L''.

$$\therefore \varepsilon_{longitudinal} = \frac{L'-L}{L} \; (+)$$

and also the diameter of the rod has decreased, which represents the lateral direction.

$$\therefore \varepsilon_{lateral} = \frac{d'-d}{d} \; (-)$$

We know that,

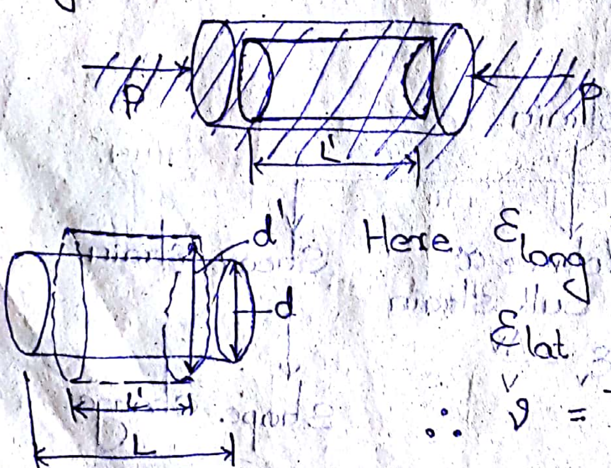
$$\text{Poisson's Ration}, \nu = \frac{1}{m} = \frac{-\varepsilon_{lat}}{\varepsilon_{long}}$$

$$\nu = \frac{d'-d}{L'-L}$$

Here, ε_{long} is a +ve quantity & ε_{lat} is a -ve quantity

Therefore, the Poisson's Ratio is a +ve term.

Similarly



Here, $\varepsilon_{long} = \frac{L'-L}{L}$ (-ve)

$\varepsilon_{lat} = \frac{d'-d}{d}$ (+ve)

$\therefore \nu = \frac{-\varepsilon_{lat}}{\varepsilon_{long}}$ is +ve

Relation between stress, strain & shear stress & shear strain:

(1) Normal

$$\sigma \propto \epsilon$$
$$\sigma = E\epsilon$$

Here E = Modulus of elasticity or Young's Modulus.
σ = Stress (Normal), ϵ = Normal strain.

(2) Shear

$$\tau \propto \gamma$$
$$\tau = G\gamma$$

Here τ = Shear stress, γ = Shear strain
G = Shear Modulus or Modulus of Rigidity

(3) Bulk or Volume

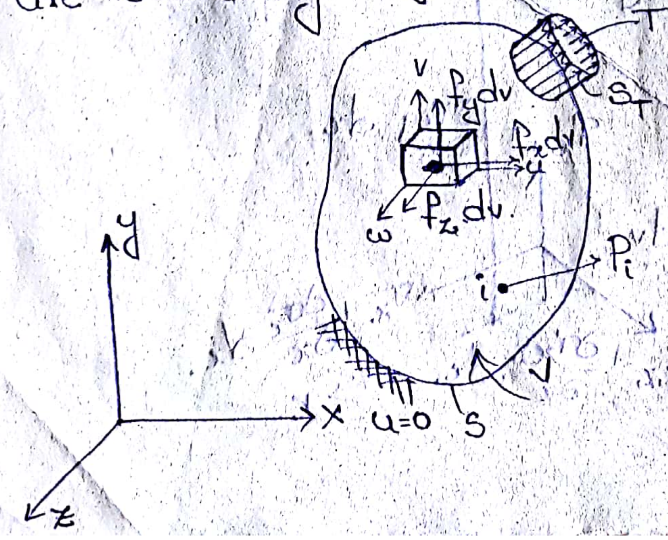
$$\sigma_v \propto \epsilon_v$$
$$\sigma_v = K\epsilon_v$$

Here σ_v = Bulk stress, ϵ_v = Bulk strain
K = Bulk Modulus.

So, Here E, G, K, & μ (Poisson's Ratio) are called elastic constants because, the values of the above doesn't change with the change in size or shape, but changes with the material.

Stresses & Equilibrium:

Consider a three-dimensional body occupying a volume 'V' and having a surface 'S' as shown. Points in the body are located by x, y, z coordinates.



On part of the boundary, distributed force per unit area, T also called Traction force is applied, under the force the body deforms.

Let the deformation at Point $= [u, v, w]^T$ at $[x, y, z]^T$

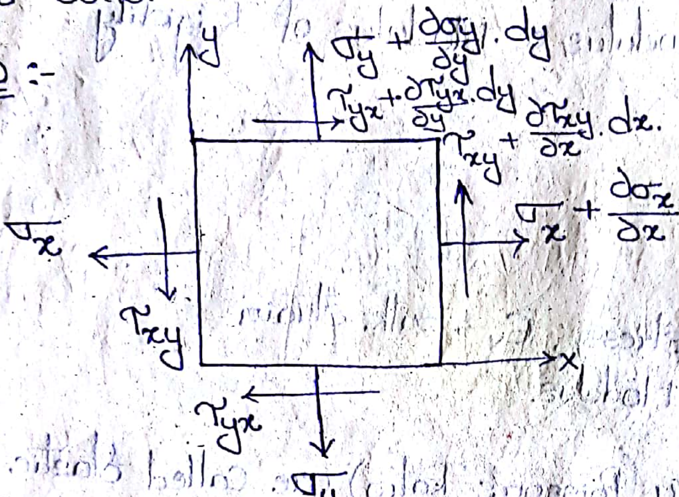
The distributed force per unit volume, $f = [f_x, f_y, f_z]^T$

The Surface Traction T is given by, $T = [T_x, T_y, T_z]^T$

A load P acting at Point 'i' is given by, $P_i = [P_x, P_y, P_z]_i^T$

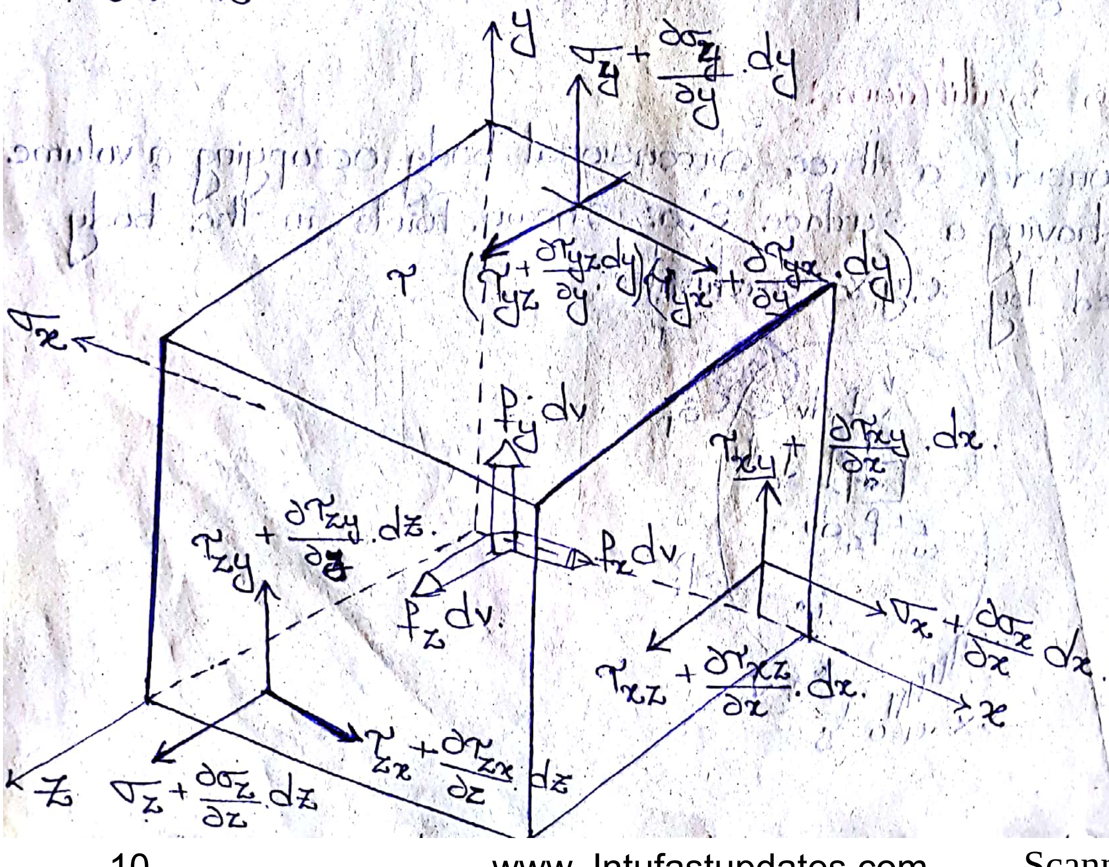
Let consider the 2D element :-

2D :-



Here $V = dx.dy$ (1)

the Same applied for 3-D : Consider the stresses acting on the elemental volume dv



Elemental volume $dv = dx.dy.dz$

Face	Stress on -ve face	Stress on +ve face
	σ_x	$\sigma_x + \frac{\partial \sigma_x}{\partial x}.dx$
x.	τ_{xy}	$\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}.dx$
	τ_{xz}	$\tau_{xz} + \frac{\partial \tau_{xz}}{\partial x}.dx$
	σ_y	$\sigma_y + \frac{\partial \sigma_y}{\partial y}.dy$
y	τ_{yx}	$\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}.dy$
	τ_{yz}	$\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y}.dy$
	σ_z	$\sigma_z + \frac{\partial \sigma_z}{\partial z}.dz$
z	τ_{zx}	$\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}.dz$
	τ_{zy}	$\tau_{zy} + \frac{\partial \tau_{zy}}{\partial z}.dz$

We have $\text{Stress} = \frac{\text{Force}}{\text{Area}}$

$\text{Force} = \text{Stress} \times \text{Area}.$

Consider the equilibrium conditions, $F_x = 0$, $F_y = 0$, & $F_z = 0$.

$$F_x = 0 \Rightarrow \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}.dx\right)dy\,dz - \sigma_x\,dy\,dz + \left(\tau_{xy} + \frac{\partial \tau_{yx}}{\partial y}.dy\right)dx\,dz$$
$$- \tau_{yx}(dx\,dz) + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}.dz\right)dx.dy - \tau_{xz}\,dx.dy + f_x\,dv = 0$$

$$\Rightarrow \cancel{\sigma_x\,dy\,dz} + \frac{\partial \sigma_x}{\partial x}.dx.dy.dz - \cancel{\sigma_x\,dy\,dz} + \cancel{\tau_{xy}\,dx\,dz} + \frac{\partial \tau_{yx}}{\partial y}.dx.dy\,dz$$
$$- \cancel{\tau_{yx}\,dx.dz} + \cancel{\tau_{zx}\,dx.dy} + \frac{\partial \tau_{zx}}{\partial z}\,dx.dy\,dz - \cancel{\tau_{xz}\,dx.dy} + f_x\,dv = 0.$$

$$\Rightarrow \frac{\partial \sigma_x}{\partial x}.dx.dy.dz + \frac{\partial \tau_{yx}}{\partial y}.dx.dy.dz + \frac{\partial \tau_{zx}}{\partial z}.dx.dy.dz + f_x\,dv = 0.$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0.$$

Similarly, we get.

$$\sum F_y = 0 \Rightarrow \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0.$$

$$\sum F_z = 0 \Rightarrow \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\tau_{yz}}{\partial y} + f_z = 0.$$

Here, $\tau_{xy} = \tau_{yx}$

$\tau_{yz} = \tau_{zy}$

$\tau_{xz} = \tau_{zx}$.

$\therefore$ The stresses acting on the body are

$$\{\sigma\} = \{\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^T$$

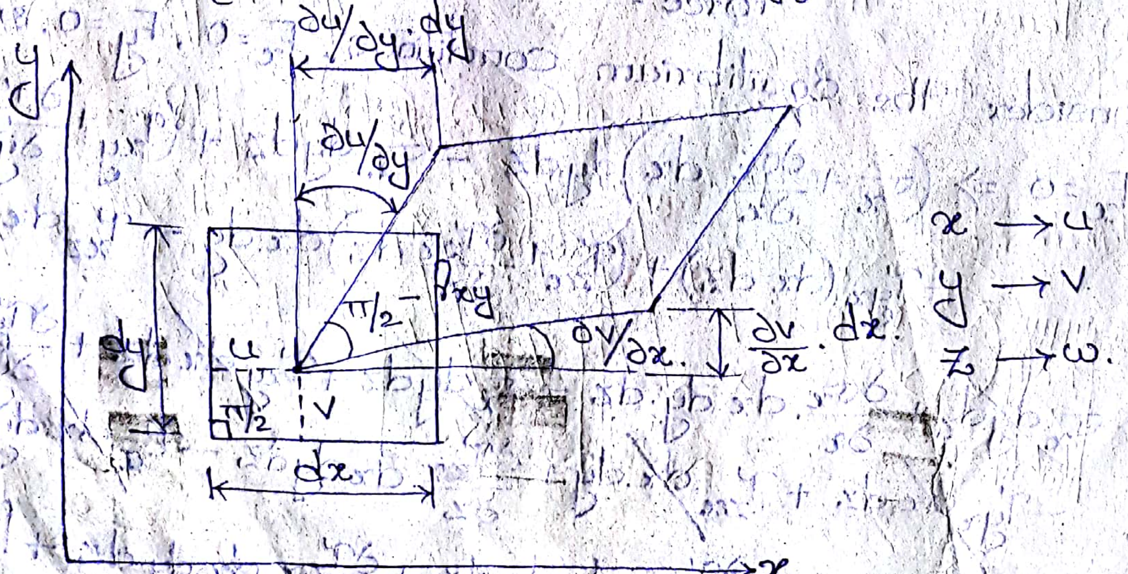
Where, $\sigma_x, \sigma_y, \sigma_z$ = Normal stresses.

$\tau_{yz}, \tau_{xz}, \tau_{xy}$ = Shear stresses.

* ~~Stress~~ Strain - Displacement Relations :

If $\epsilon_x, \epsilon_y, \epsilon_z$ are considered as Normal strains, $\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$ represents shear strains. Then these strains are are related with the displacements u, v, ω which are produced in x, y & z directions by the applied stresses.

$$\epsilon = \{\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^T$$

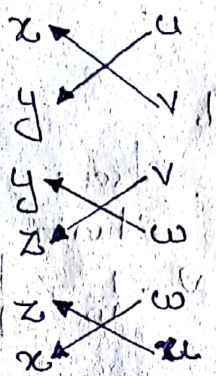


$$\epsilon_x = \frac{u}{x} = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{v}{y} = \frac{\partial v}{\partial y} \qquad \epsilon_z = \frac{\omega}{z} = \frac{\partial \omega}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$



$$\{\epsilon\} = \left\{\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}^T$$

* Stress - Strain Relations :

→ **One Dimension :**

In One dimension, we have Normal stress σ along x and and the corresponding normal-strain i.e. ϵ.

Stress - strain relation, $\sigma = E\epsilon$.

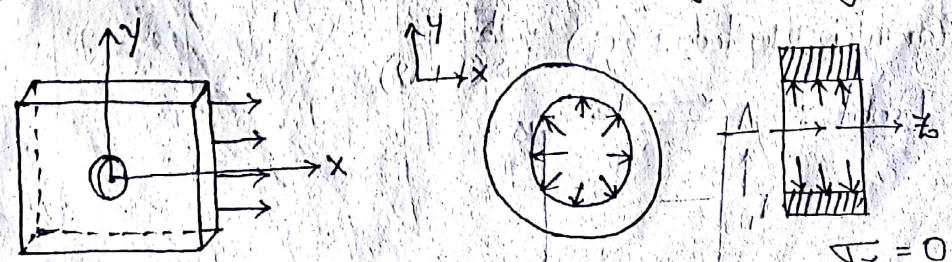
→ **Two Dimensions :**

In two dimensions, the problems are modeled as plane stress & Plane strain.

Plane stress :

A state of Plane stress is said to exist when the elastic body is very thin and there are no loads applied in the coordinate direction parallel to the thickness. In other words, for some two dimensional objects the stresses can be produced only in two directions and not possible in the third direction.

Plane stress analysis includes problems such as Plates with holes, fillets or other changes in geometry.



$\sigma_z = 0$
$\tau_{yz} = 0$
$\tau_{zx} = 0.$

Equations :-

We know that

$$E = \frac{\text{Stress}}{\text{Strain}} \quad \text{or} \quad \text{Strain} = \frac{\text{Stress}}{E}$$

& Poisson's Ratio $\nu = \frac{1}{m} = \frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \frac{\epsilon_{lat}}{\epsilon_{long}}$

$$\epsilon_{lat} = \frac{\epsilon_{long}}{m} \text{ or } \nu\,\epsilon_{long}$$

$$= \nu \frac{\text{Stress}}{E}$$

$$= \frac{\nu\sigma}{E}$$

$\therefore$ Strain in x-direction

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

Strain in Y-direction : & Z-direction

$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \qquad \epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + 0$$

& We have

$$\tau_{xy} \propto \gamma_{xy}$$

$$\tau_{xy} = G\,\gamma_{xy}$$

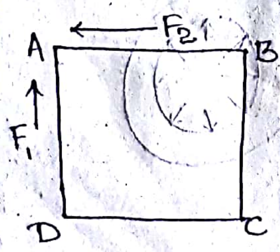
$$\therefore \gamma_{xy} = \frac{\tau_{xy}}{G}$$

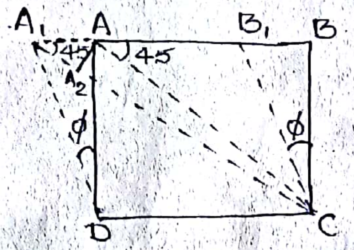
$$\gamma_{xy} = \tau_{xy} \times \left(\frac{E}{2(1+\nu)}\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\,\tau_{xy} \qquad \therefore G = \frac{E}{2(1+\nu)}$$

Rigidity Modulus (G) : $\frac{E}{2(1+\nu)}$

Proof :-





Let $AB = BC = CD = DA = a$.

Strain of $AC = \frac{\sigma}{E} + \frac{\sigma}{mE} = \frac{\sigma}{E}\left(1+\frac{1}{m}\right) = \frac{\sigma}{E}(1+\nu)$

Increase in Length, $A_1A_2 = A_1C - AC$

Here AA_2 is perpendicular to A_1C & $\angle ACA_2$ is very small and assumed as Negligible.

So $AC = A_2C$

Consider the Right angle Δle AA_1A_2

$$\cos 45 = \frac{A_1A_2}{A_1A}$$

$$A_1A_2 = \frac{A_1A}{\sqrt{2}}$$

$$\text{Shear strain } (\gamma) = \frac{\text{Transverse displacement}}{\text{Original length}}$$

$$\gamma = \frac{AA_1}{AB} = \frac{AA_1}{a}$$

$$AA_1 = \gamma a$$

$$\Rightarrow A_1A_2 = \frac{\gamma a}{\sqrt{2}}$$

$$\text{Strain of } AC = \frac{\text{Increase in length}}{\text{Original length}}$$

$$= \frac{A_1A_2}{AC}$$

We have $\cos 45 = \frac{AD}{AC} = \frac{1}{\sqrt{2}} \Rightarrow AC = a\sqrt{2}$

$$= \frac{\gamma \cancel{a}}{\sqrt{2}} \times \frac{1}{\cancel{a}\sqrt{2}}$$

$$= \frac{\gamma}{2}$$

Strain at AC $= \frac{\sigma}{E}(1+\nu)$

$$\frac{\gamma}{2} = \frac{\sigma}{E}(1+\nu)$$

$$\frac{\cancel{\sigma}}{2G} = \frac{\cancel{\sigma}}{E}(1+\nu)$$

$$G = \frac{E}{2(1+\nu)}$$

Now Coming to the stress - strain Relation for plane stress

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

Formation of Matrix

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & 0\\ -\frac{\nu}{E} & \frac{1}{E} & 0\\ 0 & 0 & \frac{2(1+\nu)}{E}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}$$

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & 0\\ -\frac{\nu}{E} & \frac{1}{E} & 0\\ 0 & 0 & \frac{2(1+\nu)}{E}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{Bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix}$$

$$\therefore \begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = E\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}^{-1}\begin{bmatrix}\epsilon_x\\ \epsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$|A| = 2(1+\nu) + \nu\left(-2\nu(1+\nu)\right)$$

$$= (1+\nu)2\left[1 - \nu^2\right]$$

$$= 2(1+\nu)(1-\nu^2)$$

$$\text{Cofactor of } A = \begin{bmatrix} 2(1+\nu) & 2\nu(1+\nu) & 0 \\ 2\nu(1+\nu) & 2(1+\nu) & 0 \\ 0 & 0 & (1-\nu^2) \end{bmatrix}$$

$$= 2(1+\nu)\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

$$\therefore A^{-1} = \frac{C^T}{|A|} = \frac{\cancel{2(1+\nu)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}}{\cancel{2(1+\nu)}(1-\nu^2)}$$

$$A^{-1} = \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$\therefore \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix}$$

$$\sigma = D\epsilon$$

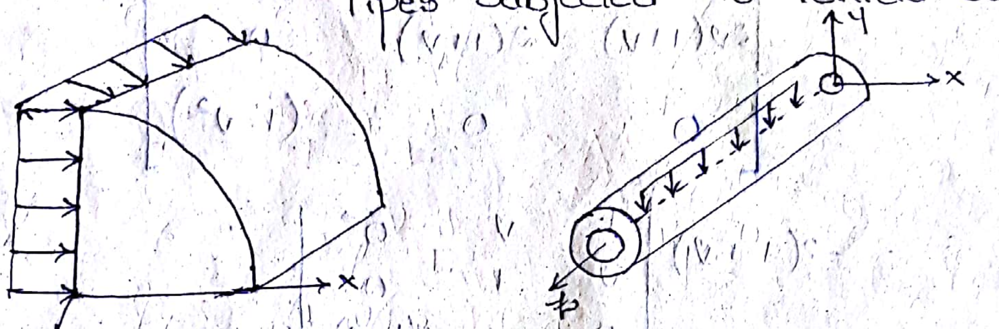
D = Stress - strain Relationship.

$$\Rightarrow D = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane Strain :

The state of plane strain occurs in members that are not free to expand in the direction perpendicular to the plane of applied forces / loads. Plane strain condition refers the occurrence of strain in the body in two directions only and in the third direction the strain is negligible & equal to Zero.

Examples :- Dams subjected to horizontal loading.
Pipes subjected to Verticle loading.



or

$\epsilon_z = 0$

$\gamma_{yz} = 0$

$\gamma_{zx} = 0$.

Here

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \left(\frac{1-2\nu}{2}\right) \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\}$$

$$D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$ = Constitutive Matrix for Plane-strain condition.

* Three Dimensional :

For Linear elastic Materials, the stress - strain relations come from the generalized Hooke's law. The properties of Materials are Young's Modulus 'E', Modulus of Rigidity 'G', & Poisson's Ratio 'ν'.

Now, Consider the elementary cube 'dv'. We get

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \longrightarrow (i)$$

$$\epsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} \longrightarrow (ii)$$

$$\epsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} \longrightarrow (iii)$$

are the Normal strain Equations &

We have, $\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{2(1+\nu)}{E}\tau_{xy} \longrightarrow (iv) \quad \because G = \frac{E}{2(1+\nu)}$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz} \longrightarrow (v)$$

$$\gamma_{xz} = \frac{2(1+\nu)}{E}\tau_{xz} \longrightarrow (vi)$$

Now, by adding the Normal strain Equation, we get

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} + \nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

$$= \frac{\sigma_x}{E}(1-\nu-\nu) + \frac{\sigma_y}{E}(1-\nu-\nu) + \frac{\sigma_z}{E}(1-\nu-\nu).$$

$$\epsilon_x + \epsilon_y + \epsilon_z = \frac{(1-2\nu)}{E}\left[\sigma_x + \sigma_y + \sigma_z\right] \longrightarrow (vii)$$

From, this we can write

$$\sigma_y + \sigma_z = \frac{E(\epsilon_x + \epsilon_y + \epsilon_z)}{(1-2\nu)} - \sigma_x \longrightarrow (viii)$$

Substituting Eq. (viii) in Eq (i), We get

$$\epsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$= \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$= \frac{\sigma_x}{E} - \frac{\nu}{E}\left[\frac{E(\epsilon_x + \epsilon_y + \epsilon_z)}{(1-2\nu)} - \sigma_x\right]$$

$$E\epsilon_x = \sigma_x - \nu\left[\frac{E(\epsilon_x + \epsilon_y + \epsilon_z)}{(1-2\nu)}\right] + \frac{\nu\sigma_x\cancel{(1-2\nu)}}{\cancel{(1-2\nu)}}$$

$$E\epsilon_x + \nu \frac{E(\epsilon_x+\epsilon_y+\epsilon_z)}{(1-2\nu)} = \sigma_x + \nu\sigma_x$$

$$E\epsilon_x(1-2\nu) + \nu E\epsilon_x + \nu E\epsilon_y + \nu E\epsilon_z = \sigma_x(1+\nu)(1-2\nu)$$

$$E\left(\epsilon_x - 2\nu\epsilon_x + \nu\epsilon_x + \nu\epsilon_y + \nu\epsilon_z\right) = \sigma_x(1+\nu)(1-2\nu)$$

$$E\left(\epsilon_x - \nu\epsilon_x + \nu\epsilon_y + \nu\epsilon_z\right) = \sigma_x(1+\nu)(1-2\nu)$$

$$E\left(\epsilon_x(1-\nu) + \nu\epsilon_y + \nu\epsilon_z\right) = \sigma_x(1+\nu)(1-2\nu)$$

$$\therefore \sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[\epsilon_x(1-\nu) + \nu\epsilon_y + \nu\epsilon_z\right] \longrightarrow (ix)$$

Similarly, by substituting

$$\sigma_x + \sigma_z = \frac{E(\epsilon_x+\epsilon_y+\epsilon_z)}{(1-2\nu)} - \sigma_y \quad \text{in Eq (ii)}$$

&

$$\sigma_x + \sigma_y = \frac{E(\epsilon_x+\epsilon_y+\epsilon_z)}{(1-2\nu)} - \sigma_z \quad \text{in Eq (iii)}$$

We get,

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[\epsilon_x\nu + (1-\nu)\epsilon_y + \nu\epsilon_z\right] \longrightarrow (x)$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[\epsilon_x\nu + \nu\epsilon_y + (1-\nu)\epsilon_z\right] \longrightarrow (xi)$$

& the Normal stress can be written as

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{2}\right)\gamma_{xy} \longrightarrow (xii)$$

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} = \frac{E}{(1+\nu)(1-2\nu)}\frac{(1-2\nu)}{2}\gamma_{yz} \longrightarrow (xiii)$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\gamma_{xz} = \frac{E}{(1+\nu)(1-2\nu)}\frac{(1-2\nu)}{2}\gamma_{xz} \longrightarrow (xiv)$$

from the Eq. (ix), (x), (xi), (xii), (xiii) & (xiv), We get the inverse relations

$$\sigma = D\epsilon$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{zy} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix}$$

Here,

D = Stress - strain Relationship Matrix

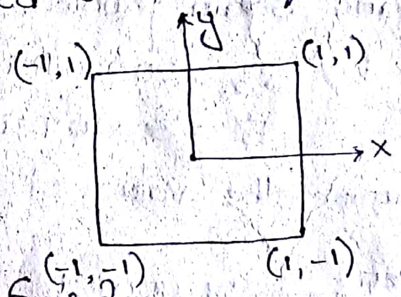
$$= \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5-\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5-\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Problems :-

1. A displacement field is imposed on the square element shown in fig.

$u = 1 + 3x + 4x^3 + 6xy^2$

$v = xy - 7x^2$



(a) Write the expressions for ϵ_x, ϵ_y & γ_{xy}.

(b) Find where ϵ_x is Maximum.

Sol: Given $u = 1 + 3x + 4x^3 + 6xy^2$

$v = xy - 7x^2$

A(-1, -1) B(1, -1) C(1, 1) D(-1, 1).

(a) Expressions for $\epsilon_x, \epsilon_y, \gamma_{xy}$

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(1 + 3x + 4x^3 + 6xy^2\right)$$

$$\epsilon_x = 3 + 12x^2 + 6y^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(xy - 7x^2)$$

$$\epsilon_y = x$$

8) $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

$= \frac{\partial}{\partial y}\left(1+3x+4x^3+6xy^2\right) + \frac{\partial}{\partial x}\left(xy-7x^2\right)$

$\gamma_{xy} = 12xy + y - 14x.$

(b) Find where ϵ_x is Maximum.

$\epsilon_x = 3+12x^2+6y^2$

Let $A(-1,-1) \Rightarrow \epsilon_x = 3+12(-1)^2+6(-1)^2 = 21.$

$B(1,-1) \Rightarrow \epsilon_x = 3+12+6 = 21$

$C(+1,+1) \Rightarrow \epsilon_x = 3+12+6 = 21$

$D(-1,1) \Rightarrow \epsilon_x = 21.$

At $(0.5,0.5) \Rightarrow \epsilon_x = 7.5$

$(-1,-0.5) \Rightarrow \epsilon_x = 16.5$

It is observed that ϵ_x is Maximum at the edges.

2. In a plane strain problem, we have

$\sigma_x = 20{,}000 \text{ Psi}$, $\sigma_y = -10{,}000 \text{ Psi}$

$E = 30\times10^6 \text{ Psi}$, $\mu = 0.3.$

Determine the value of stress at σ_z.

<u>Sol</u> Given $\sigma_x = 20{,}000 \text{ Psi}$ $\sigma_y = -10{,}000 \text{ Psi}$

$E = 30\times10^6 \text{ Psi}$

from stress - strain relation.

$$\epsilon_z = \frac{-\mu\sigma_x}{E} - \mu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

for plain strain $\epsilon_z = 0.$

$$-\frac{\mu\sigma_x}{E} - \mu\frac{\sigma_y}{E} + \frac{\sigma_z}{E} = 0.$$

$$\sigma_z = \mu(\sigma_x+\sigma_y)$$

$$= 0.3(20{,}000-10{,}000)$$

$$= 0.3(10{,}000).$$

$$\sigma_z = 3000 \text{ Psi}$$

* Functional Approximation Methods :

These methods are adopted for finding approximate solutions mostly for the complex problems like Non-linear & continuous systems in the field of Solid Mechanics. In these Methods, the ~~py~~ physical problems by which the solution is to be found out are first written in the form of Suitable Governing Equations (Differential Eq's) or any possible mathematical expressions. By integrating & applying boundary conditions, the required approximate solution can be determined.

The nature of problems for which the solution is to be found out are usually specified in three types such as

1. Equilibrium (Equations) Problems.
2. Eigen Value problems
3. Probagation problems.

The Functional Apporximation methods for solving the above types of problems are classified into two major types. such as.

1. Variational Methods → Rayleigh Ritz Method.
2. Weighted Residual Methods.
   1. Point Collocation
   2. Sub-domain Collocation
   3. Least Square Method.
   4. Galerkin's Method.

1. Variational Method :

In Variational Method, the physical problem expressed in terms of differential equations is recast in an Equivalent integral form. Then with the help of trial functions, this integral, called as functional, is made to reach to Extremum conditions such as Maximum or Minimum conditions, it is said to be stationary. The trial function which makes the integral to attain the stationary value is the required approximate solution for problem.

The term 'Variational Methods' refers to the methods that make use of Variational principles such as the principle of Virtual work and the principle of Minimum Potential Energy in Solid & Structural Mechanics to find the approximate solution.

* **Concept of Potential Energy :-**

Let consider an elastic system subjected to External forces, it will have some deformation ie displacement. During displacement of an elastic system, External work as well as internal work are involved. A deformed elastic bodies is said to possess two kinds of potential energies. The energy arising due to the workdone by the External forces and the energy stored within the body as Strain Energy. These two energies combined together constitute the potential energy of the system.

→ **Constituents of Total Potential Energy :**

For a Rigid system, the total potential energy, denoted by π, is due to the external forces alone. But in the case of deformable (i.e., elastic) system, the total potential energy π is due to

(i) External force

(ii) Strain Energy.

The Work-done due to External force may be either positive or negative depending upon how the forces & displacement acts. And the strain energy, alternatively known as 'Elastic potential Energy' is always a Positive quantity.

Mathematically,

$$\pi = U \pm W.$$

where U = Strain Energy

W = Workdone due to External forces.

Note :- On the System = −ve W.D.

By the System = +ve W.D.

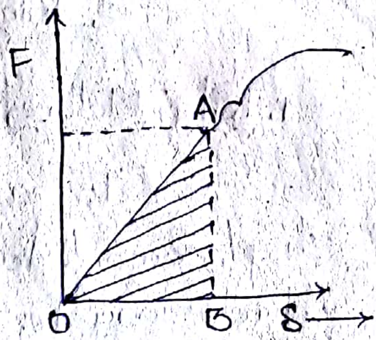
Here,

Strain Energy = The amount of Energy stored in a body, due to the workdone by the force applied on it, within the elastic limit.

U = Area under the stress-strain curve upto elastic limit.

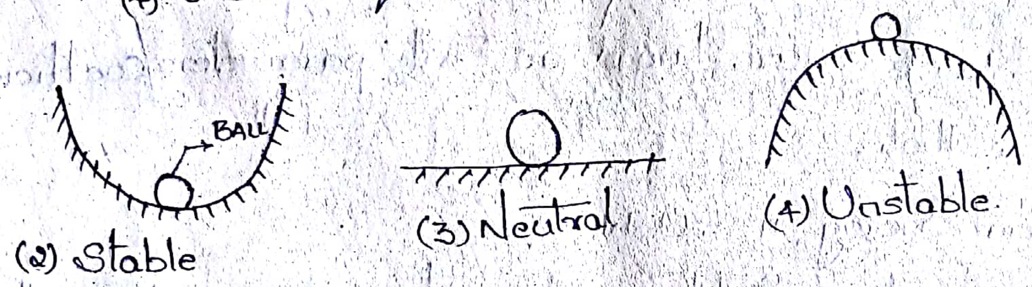
= Area of ΔOAB

$= \frac{1}{2} F\delta$



Variation characteristics of Total potential Energy :-

(1) Static Equilibrium = π is stationary.
(2) Stable Equilibrium = π is Minimum (M.P.E)
(3) Neutral Equilibrium = Unchanged π
(4) Unstable Equilibrium = π is Maximum



→ **Principle of Minimum potential Energy** : If the Extremum condition is Minimum, the Equilibrium state is stable.

→ **Principle of Virtual Work** :

The principle of virtual work is another formulative procedure to find the approximate solution for FAM. According to the principle, a body is in Equilibrium under the action of External forces if the External virtual work for every kinematically admissible displacement is equal to the internal virtual strain Energy.

$$\Delta W = \Delta U$$

where ΔW = External Virtual work.
ΔU = Internal Virtual strain Energy.

* Rayleigh - Ritz Method :

In this method the problems are solved in two ways. They are.

1. Minimum Potential Energy Method.
2. Integral Approach Method.

1. Ritz Method based on Minimum Potential Energy Method :

To solve a problem in Ritz method, at first a displacement function is assumed in terms of Ritz coefficients & which may be from one to infinity.

Let $y(x)$ be the displacement function.

$y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \dots$ (Polynomial Series)

or

$y(x) = a_1 \sin \frac{\pi x}{L} + a_2 \sin \frac{3\pi x}{L} + a_3 \sin \frac{5\pi x}{L} + \dots$ (Trigonometric Series)

Here $a_1, a_2, a_3 \dots$ are known as Ritz parameters/Coefficients.

We know that,

Total Potential Energy, $\pi = U - W$.

In structural Problems, the strain Energy U & the External W.D can be specified as a functions of displacement $y(x)$. By making the total potential energy to reach minimum, the approximate solution can be determined.

In general, for Minimum potential Energy.

$$\frac{\partial \pi}{\partial a_1} = 0 , \frac{\partial \pi}{\partial a_2} = 0 , \frac{\partial \pi}{\partial a_3} = 0 , \dots \frac{\partial \pi}{\partial a_n} = 0 \quad ; n = 1 \text{ to } \infty$$

The Accuracy depends on no. of Ritz Coefficients.

2. Ritz Method based on Integral Approach :

In this case, the physical Problem expressed in terms of differential equation is recast in an Equivalent integral form. With the help of trial function, this integral is made to reach the Extremum condition such as Max. or Min.

For Example, Consider a physical problem in terms of differential Equations as

$$D\frac{d^2y}{dx^2} + Q = 0 \quad , \text{B.C } y(0) = y_0 \ \& \ y(l) = y_l.$$

$y = 0$ at $x = 0$
$y = y_l$ at $x = l$.

Here D & Q are Constants

The above D.E can be written as

$$I = \int_0^L \left[\frac{D}{2}\left(\frac{dy}{dx}\right)^2 - Qy\right] dx.$$

I is termed as the Functional.

The trial function is selected from polynomial Series or trigonometric series as expressed in Equations above that is

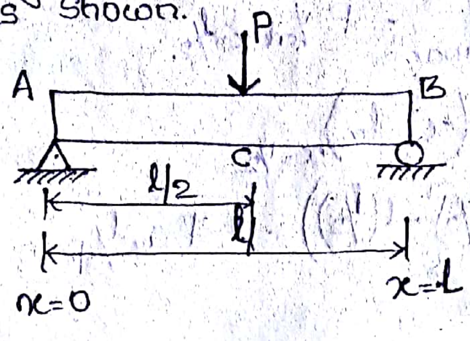
$$y(x) = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots\cdots$$

The selected trial is differentiated suitably & the value of functional I can be determined in terms of Ritz coefficients. Then, the functional is made to reach the constant value. for stationary value of functional, the following Conditions must be satisfied.

i.e., $\frac{\partial I}{\partial a_1} = 0$, $\frac{\partial I}{\partial a_2} = 0$, $\frac{\partial I}{\partial a_3} = 0$, - - - - $\frac{\partial I}{\partial a_n} = 0$, $n = 1$ to ∞

Problems :-

1. Find the deflection at the centre of a simply supported beam of length l subjected to a concentrated load P at its mid-point as shown.



Sol Consider,

$\pi = U - W$

U = Strain energy

W = Workdone by external force.

Strain energy for a beam,

$$U = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx.$$

Where E = Modulus of elasticity

I = Area M.I

$y = a_1 + a_2x + a_3x^2 + a_4x^3 + \ldots\ldots$

To simplify the problem, Consider

$y = a_1 + a_2x + a_3x^2$

The boundary conditions are $y=0$ at $x=0$ and $x=l$.

$\therefore y = a_1 = 0$.

& $0 = a_1 + a_2l + a_3l^2 \Rightarrow a_2l + a_3l^2 = 0$

$a_2l = -a_3l^2 \Rightarrow a_2 = -a_3l$.

$\therefore y$ can be expressed as

$y = 0 - a_3lx + a_3x^2 = a_3(x^2 - lx)$

$\frac{dy}{dx} = a_3(2x - l)$

$\frac{d^2y}{dx^2} = 2a_3$

$$\therefore U = \frac{EI}{2}\int_0^l (2a_3)^2 dx = \frac{EI}{2}4a_3^2 \cdot l = 2EIa_3^2 \times l.$$

Now,

Work-done $W = P \times y_{max} = P\, y_{at\, x = l/2}$

$= Pa_3(x^2 - lx)_{x=l/2}$

$= Pa_3\left((l/2)^2 - l(l/2)\right) = -Pa_3 l^2/4$.

$\therefore$ Total Potential Energy, $\pi = U - W$

$$= 2EI\, a_3^2 l - \left(-Pa_3 \frac{l^2}{4}\right)$$

$$= 2EI a_3^2 l + Pa_3 \frac{l^2}{4}$$

For Minimum potential Energy Condition, $\frac{\partial \pi}{\partial a_3} = 0$.

$$\frac{\partial \pi}{\partial a_3} = 0 \Rightarrow 4EIa_3 l + \frac{Pl^2}{4} = 0$$

$$4EIa_3 l = -\frac{Pl^2}{4}$$

$$a_3 = -\frac{Pl}{16EI}$$

Substituting a_3 in $y = a_3(x^2 - lx)$

$$= -\frac{Pl}{16EI}(x^2 - lx)$$

Maximum deflection occurs at $x = l/2$

$$\therefore y_{max} = -\frac{Pl}{16EI}\left(\frac{l^2}{2^2} - l\left(\frac{l}{2}\right)\right)$$

$$= -\frac{Pl}{16EI}\left(\frac{l^2}{4} - \frac{l^2}{2}\right)$$

$y_{max} = \frac{Pl^3}{64EI}$ which is an Approximate Solution.

But the Exact solution for the Max. deflection for a Simply supported beam subjected to point load at centre is

$$y_{max} = \frac{Pl^3}{48EI}$$

So, to get more accurate results, the displacement function should contain more number of Ritz Coefficients, that is

$$y = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \ldots$$

3. Consider the differential equation for a Problem such as $\frac{d^2y}{dx^2} + 300x^2 = 0$; $0 \le x \le 1$ with the b.c $y(0) = y(1) = 0$. The functional corresponding to this problem to be extremized is given by $I = \int_0^1 \left\{ -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 + 300x^2 y \right\} dx$.

Find the solution using Rayleigh-Ritz method using a one term solution as $y = ax(1-x^3)$.

Sol: Given $y = ax(1-x^3)$

$y = ax - ax^4$, it satisfies $y = 0$ at $x = 0$ & $x = 1$.

Now differentiating the above Eq., we get

$$\frac{dy}{dx} = a - 4ax^3$$

$$\therefore I = \int_0^1 \left\{ -\frac{1}{2}(a - 4ax^3)^2 + 300x^2(ax - ax^4) \right\} dx$$

$$= \int_0^1 \left\{ -\frac{1}{2}(a^2 + 16a^2x^6 - 8a^2x^3) + (300x^3a - 300ax^6) \right\} dx$$

$$= -\frac{1}{2}\left[a^2x + 16a^2\frac{x^7}{7} - 8a^2\frac{x^4}{4} \right]_0^1 + \left[300\frac{x^4}{4}a - 300a\frac{x^7}{7} \right]_0^1$$

$$= -\frac{1}{2}\left[a^2 + \frac{16}{7}a^2 - \frac{8}{4}a^2 \right] + \left[\frac{300a}{4} - \frac{300a}{7} \right]$$

$$I = \frac{-a^2}{2} - \frac{8}{7}a^2 + a^2 + \frac{300}{4}a - \frac{300}{7}a$$

Now, for attaining a stationary value, $\frac{\partial I}{\partial a} = 0$.

$$\frac{\partial I}{\partial a} = 0 \Rightarrow -a - \frac{16a}{7} + 2a + \frac{300}{4} - \frac{300}{7} = 0$$

By solving, we get $a = 25$.

Hence $y = 25x(1-x^3)$.

* Weighted Residual Method :

This Method is employed to obtain approximate solution for linear or non-linear mostly for Non-structural problems whose characteristics expressed in terms of differential Equations.

In Majority of the problems, getting the exact solution seems highly difficult and we get the approximate solution, containing some errors. If this error is minimized by some way, then the approximate solution will be almost equal to the Exact solution. In Weighted Residual Methods, efforts have been taken to minimize the errors by some kind of procedures. Based on the Methods of Minimizing the error from approximate solution, the weighted residual technique can be adopted in four methods such as.

1. Point Collocation Method.
2. Sub-domain Collocation Method.
3. Least Square Method.
4. Galerkin's Method.

Let, Consider $y(x)$ is the Exact solution for the D.E. Suppose the Exact solution can not be found out, then other approximate function called trial function $y(x) = f(x, a_i)$, $i = 1, 2, \ldots$ must be considered and this trial function is substituted in the differential Equation & the residual $R(x, a_i)$ will be found out. This residual is equated to zero directly or by combining with other parameters and thus the required solution is obtained.

$$\therefore \int W_i R(x, a_i)\, dx = 0, \quad (i = 1, 2, \ldots n)$$

The No. of weight functions is equal to the no. of Unknown coefficients in the approximate function.

1. Point Collocation Method :

In this method, the Residual $R(x;a_i)$ is set equal to Zero, at n specific points $x_1, x_2, x_3, \dots x_n$.

$$W_i = \delta(x - x_i)$$

$$\therefore \int_D \delta(x - x_i) R(x, a_i)\, dx = 0.$$

At Point $x = x_i$, $W_i = 1$ & hence $R(x, a_i) = 0$, and at other points in the domain $W_i = 0$.

2. Sub-domain Collocation Method :

In this Method, the domain is subdivided into n sub-domains and the integral of the residual over each sub-domain is then required to be Zero.

$$\int_D R(x, a_i)\, dx = 0.$$

3. Least Squares Method :

$$I = \int_D [R(x, a_i)]^2 dx = \text{Minimum.}$$

$$\frac{\partial I}{\partial a_i} = 0, \quad i = 1, 2, \dots n.$$

4. Galerkin's Method :

$$\int_D W_i R(x, a_i)\, dx = \int_D y(x) R(x, a_i)\, dx = 0$$

Here the trial function $y(x)$ itself is considered as the weighting function.

1. Solve the differential equations for a physical problem expressed as
$$\frac{d^2y}{dx^2} + 100 = 0 , 0 \le x \le 10.$$
with b.c as $y(0)=0$ & $y(10)=0$ using (i) Point Collocation Method (ii) Sub-domain Collocation Method, (iii) Least Squares & (iv) Galerkin's Method.

Sol: The D.E is
$$\frac{d^2y}{dx^2} + 100 = 0 , \quad x \text{ ranges from } 0 \text{ to } 10.$$
B.C are $y=0$ at $x=0$ & $x=10$.

Now, Assume a trial function for D.E which should satisfy the B.C also.
$$y = ax(x-10)$$
$$y = a(x^2-10x)$$
$$\frac{dy}{dx} = a(2x-10) \quad \& \quad \frac{d^2y}{dx^2} = 2a.$$
$$\therefore R = 2a+100$$

(i) Point Collocation Method:

Here, the residual is set to zero.
$$R = 2a+100 = 0$$
$$a = -50$$
Hence the traial function is $y = -50x(x-10)$
$$y = 50x(10-x).$$

(ii) Sub-domain Collocation Method:

Here, the integral of residual over the sub-domain is set to zero.
$$\int_0^{10} R\,dx = 0$$
$$\int_0^{10} (2a+100)\,dx = 0.$$
$$\left[2ax + 100x\right]_0^{10} = 0.$$

$20a + 1000 = 0$

$a = -50.$

Hence the trial function $y = -50x(x-10) = 50x(10-x)$

(iii) Least Square Method :

Here, the integral of the square of the residual over the domain is to be Minimum.

$$I = \int_0^{10} R^2 dx = \text{Min}.$$

$$= \int_0^{10} R^2 dx = \int_0^{10} (2a+100)^2 dx.$$

$$= \int_0^{10} (4a^2 + 400a + 10000) dx$$

$$= \left[4a^2x + 400ax + 1000(x)\right]_0^{10}$$

$$= 40a^2 + 4000a + 100000.$$

$$\frac{\partial I}{\partial a} = 0 \Rightarrow 80a + 4000 = 0.$$

$$a = -50.$$

$$\therefore y = -50x(x-10) = 50x(10-x)$$

(iv) Galerkin's Method:

Here, the domain integral of the product of the trial function with the residual is set to zero.

$$\Rightarrow \int_0^{10} (y \cdot R) . dx = 0.$$

$$\Rightarrow \int_0^{10} ax(x-10)(2a+100) dx = 0.$$

$$\Rightarrow \int_0^{10} (2a^2x^2 + 100ax^2 - 20a^2x - 1000ax) dx = 0.$$

$$\Rightarrow \left[2a^2\frac{x^3}{3} + 100a\frac{x^3}{3} - 20a^2\frac{x^2}{2} - 1000\,a\frac{x^2}{2}\right]_0^{10} = 0.$$

$$\Rightarrow \frac{2}{3}a^2(1000) + \frac{100}{3}a(1000) - 10a^2(100) - 500a(100) = 0.$$

$$\Rightarrow \frac{2}{3}a + \frac{100}{3} - a - 50 = 0$$

$$\Rightarrow a = -50$$

$$\therefore y = -50x(x-10) = 50x(10-x)$$